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Chaplygin's Transformation Applied to Magnetogasdynamics

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The studies[1–6] concerning two-dimensional magnetogasdynamics with infinite conductivity and either an aligned or a transverse magnetic field seem to have missed the fact that all of their exact solutions can be derived more easily by applying the fundamental relations developed by Bers and Gelbart[7] and, even more important, the fact that the analytic continuation through the sonic line should be expressed in terms of the Chaplygin[8] transformation σ rather than the canonical transformation S introduced by Christianovich.[9]

Following Bers and Gelbart,[7] we note that if we can write the following expressions in the physical (x, y) plane

$$Q(q)\cos\theta = \partial\phi/\partial x = P(q)(\partial\psi/\partial y)$$
$$Q(q)\sin\theta = \partial\phi/\partial y = -P(q)(\partial\psi/\partial x) \tag{1}$$

where P and Q are analytic functions of the velocity magnitude (q) only, then a solution in the hodograph (q, θ) plane may be written as

$$\psi(q, \theta) = \sum_{m=0}^{\infty} C_m\psi_m(q, \theta)$$
$$\psi_{n+1} = \int \psi_n d\theta + \int (PQ)^{-1}\phi_n dQ \tag{2}$$
$$\phi_{n+1} = \int \phi_n d\theta + \int Q\,\frac{d(P/Q)}{dQ}\,\psi_n dQ$$

where ψ_n and ϕ_n represent any solutions of the generalized hodograph equations

$$\frac{\partial\phi}{\partial\theta} = PQ\frac{\partial\psi}{\partial Q} \qquad \frac{\partial\phi}{\partial Q} = Q\,\frac{d(P/Q)}{dQ}\,\frac{\partial\psi}{\partial\theta} \tag{3}$$

For example, taking $\psi_0 = 0$ and $\phi_0 =$ const means that ψ_1 corresponds to a generalized vortex flow, whereas $\phi_0 = 0$ and $\psi_0 =$ const yield the equivalent of a source or a sink. Then a superposition of the two solutions gives a spiral type flow, and so on. The flow about an infinite half-plane, corresponding to the Ringleb[10] solution, is given by

$$\psi = CQ^{-1}\sin\theta \qquad \phi = CPQ^{-1}\cos\theta \tag{4}$$

In this same general form the Chaplygin[8] transformation may be written as

$$\frac{d\sigma}{dQ} = -\frac{1}{PQ} \qquad \frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial\sigma} \qquad \frac{\partial\phi}{\partial\sigma} = K\frac{\partial\psi}{\partial\theta}$$
$$\frac{\partial^2\psi}{\partial\sigma^2} + K\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{5}$$
$$K(\sigma) = -PQ^2\,\frac{d(P/Q)}{dQ}$$

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On the other hand, the canonical transformation becomes

$$\frac{dS}{dQ} = -\frac{K^{1/2}}{PQ} \qquad \frac{\partial\phi}{\partial\theta} = -K^{1/2}\frac{\partial\psi}{\partial S}$$
$$\frac{\partial\phi}{\partial S} = +K^{1/2}\frac{\partial\psi}{\partial\theta} \qquad \frac{\partial^2\psi}{\partial S^2} + \frac{\partial^2\psi}{\partial\theta^2} = -\frac{1}{K^{1/2}}\frac{dK^{1/2}}{dS}\frac{\partial\psi}{\partial S} \tag{6}$$

The well-known relations for isentropic perfect gas flow are immediately given by writing

$$\left.\begin{aligned} &Q = q \qquad P = \frac{\rho_0}{\rho_0} \qquad \frac{d(P/Q)}{dQ} = -\frac{\rho_0}{\rho q^2}(1 - M^2) \\ &K = -\frac{\rho_0 q^2}{\rho}\frac{d(\rho_0/\rho q)}{dq} = \left(\frac{\rho_0}{\rho}\right)^2(1 - M^2) = \\ &\qquad 2\left(\frac{\gamma+1}{2}\right)^{(\gamma+2)/(\gamma-1)}\sigma + 0(\sigma^2) \\ &\frac{1}{K^{1/2}}\frac{dK^{1/2}}{dS} = \frac{\gamma+1}{2}\frac{M^4}{(1 - M^2)^{3/2}} = -\frac{1}{3S} + O(S^{-1/3}) \end{aligned}\right\} \tag{7}$$

The magnetogasdynamic flow with an aligned magnetic field, wherein the magnetic field lines are everywhere coincident with the streamlines, may be written by following Iur'ev,[2] for a gas with infinite conductivity, as

$$\mathbf{B} = \alpha\rho\mathbf{q} \qquad \mathbf{q}\times(\nabla\times\mathbf{q}) = (\rho\mu)^{-1}\mathbf{B}\times(\nabla\times\mathbf{B})$$
$$\frac{\partial}{\partial x}[v(1 - m\rho)] = \frac{\partial}{\partial y}[u(1 - m\rho)] \tag{8}$$

where $m = \alpha^2/\mu =$ const. These relations correspond to Eq. (1) if we write

$$Q = (u^2 + v^2)^{1/2}(1 - m\rho) = q(1 - m\rho)$$
$$P = (\rho_0/\rho)(1 - m\rho) = (\rho_0/\rho)(Q/q) \tag{9}$$

so that Eq. (5) reduces to

$$\frac{d\sigma}{dq} = \frac{\rho_0}{\rho q}(1 - m\rho)^{-2}[1 - m\rho(1 - M^2)]$$
$$K(\sigma) = \left(\frac{\rho_0}{\rho}\right)^2(1 - M^2)\left[\frac{(1 - m\rho)^3}{1 - m\rho(1 - M^2)}\right] = K(S) \tag{10}$$

where K is in agreement with the resuts first given by Iu'rev[2] and later by Smith.[6] The effect of a weak magnetic field is clearly shown by the series expansion

$$K(\sigma) = \left(\frac{\rho_0}{\rho}\right)^2(1 - M^2)\left[1 - \frac{\alpha^2\rho}{\mu}(2 + M^2) + 0\left(\frac{\alpha^2\rho}{\mu}\right)^2\right] \tag{11}$$

Similarly, for the vortex flow we obtain from Eqs. (3) and (9)

$$\phi = \theta \qquad \psi = \sigma \qquad (a_0/a_*)^2 = (\gamma + 1)/2$$
$$\psi = \int\frac{\rho}{\rho_0}\left[\frac{1 - m\rho(1 - M^2)}{(1 - m\rho)^2}\right]\frac{dq}{q} = \tag{12}$$
$$\left(1 + \frac{\alpha^2\rho_0}{\mu}\right)\ln\left[\left(\frac{\gamma-1}{2}\right)^{1/2}\frac{q}{a_0}\right] - \frac{1}{4}\left(\frac{q}{a_0}\right)^2 + \ldots$$

and from Eq. (4) we obtain the equivalent of the Ringleb[10] flow as

$$\phi = (\rho_0/\rho_q)\cos\theta$$
$$\psi = q^{-1}\left(1 - \frac{\alpha^2\rho}{\mu}\right)^{-1}\sin\theta = \frac{\sin\theta}{q}\left[1 + \frac{\alpha^2\rho}{\mu} + 0\left(\frac{\alpha^2\rho}{\mu}\right)^2\right] \tag{13}$$

When the same procedure is applied to the transverse magnetic field that is assumed to remain everywhere normal to the plane of the flow, we have from Nochevkina[1] or Grad,[3] for a gas with infinite conductivity,

$$\left.\begin{aligned}&\mathbf{B} = \mathbf{k}\beta\rho(x, y) \qquad b^2 = \frac{B^2}{\mu\rho} = \frac{\beta^2\rho}{\mu} \qquad a^2 = \frac{\gamma p}{\rho}\\ &\frac{q^2}{2} + \frac{a^2}{\gamma - 1} + b^2 = h_0 = \frac{a_0{}^2}{\gamma - 1} + b_0{}^2\\ &K = \left(\frac{\rho_0}{\rho}\right)^2\left[1 - M^2\left(1 + \frac{b^2}{a^2}\right)^{-1}\right] =\\ &\left(\frac{\rho_0}{\rho}\right)^2\left[\frac{\gamma+1}{\gamma-1}a^2 + 3b^2 - 2h_0\right](a^2 + b^2)^{-1}\end{aligned}\right\} \quad (14)$$

Then the vortex flow is given by

$$\psi = \int \frac{\rho}{\rho_0 q}dq = -\frac{1}{2\rho_0}\int\frac{(a^2 + b^2)d\rho}{\{h_0 - [a^2/(\gamma - 1)] - b^2\}} \quad (15)$$

whereas the equivalent of the Ringleb solution is

$$\psi = \left[2\left(h_0 - \frac{a^2}{\gamma - 1} - b^2\right)\right]^{-1/2}\sin\theta \quad (16)$$

The corresponding solutions can also be easily obtained in terms of the variable S in the canonical transformation given by Eq. (6). However, it will now be shown that the analytical continuation through the sonic line is better expressed in terms of the Chaplygin transformation σ given by Eq. (5). For an isentropic perfect gas flow we find from Eqs. (5) and (7) that we can write for the Chaplygin transformation

$$\left.\begin{aligned}&\sigma(q) = \int_q^{a_*}\frac{\rho}{\rho_0}\frac{dq}{q} = \sigma'(a_*)(q - a_*) + \frac{\sigma''(a_*)}{2}(q - a_*)^2 + \ldots\\ &\sigma'(a_*) = -\frac{1}{a_*}\left(\frac{\gamma+1}{2}\right)^{-1/(\gamma-1)}\\ &\sigma''(a_*) = \frac{2}{a_*{}^2}\left(\frac{\gamma+1}{2}\right)^{-1/(\gamma-1)}\\ &\frac{q}{a_*} = 1 - \left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)}\sigma + \left(\frac{\gamma+1}{2}\right)^{2/(\gamma-1)}\sigma^2 + 0(\sigma^3)\end{aligned}\right\} \quad (17)$$

Consequently, any proper analytic function of q can have an analytic continuation across the sonic line. On the other hand, we find from Eqs. (6) and (7) that for the canonical transformation we must write

$$S(q) = \int_q^{a_*}(1 - M^2)^{1/2}\frac{dq}{q} = \left\{\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tanh^{-1}(1 - M^2)^{1/2}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2} - \tanh^{-1}(1 - M^2)^{1/2}\right\} = -\frac{(1 - M^2)^{3/2}}{3}\left(\frac{2}{\gamma+1}\right) - \frac{\gamma(1 - M^2)^{5/2}}{5}\left(\frac{2}{\gamma+1}\right)^2 + \ldots \quad (18)$$

since $S'(a_*) = 0$ and all the higher derivatives are infinite on the sonic line. Therefore, near the sonic line we have

$$\frac{q}{a_*} = 1 - \left(\frac{1 - M^2}{\gamma + 1}\right) - \left(\frac{2\gamma - 1}{2}\right)\left(\frac{1 - M^2}{\gamma + 1}\right)^2 + 0(1 - M^2)^3 = \left[1 - \left(\frac{3}{2}\right)^{2/3}(\gamma + 1)^{-1/3}(-S)^{2/3} + \left(\frac{5 - 2\gamma}{10}\right)\left(\frac{3}{2}\right)^{4/3}(\gamma + 1)^{-2/3}(-S)^{4/3} + 0(-S)^2\right] \quad (19)$$

Consequently, all of the derivatives of q with respect to the canonical variable S are infinite on the sonic line. It is shown in Ref. 11 how much better the variable σ represents the transonic solution corresponding to the Ringleb[10] flow about a semi-infinite half-plane. Equations (11–13) show how easily series expansions can be obtained by using the variable σ or q itself. These particular series expansions are useful because they immediately give a magnitude estimate, or a correction term, for the effect of a weak magnetic field.

Finally, it should be noted that, although the assumption of an infinite conductivity has some physical significance as a limiting case for an aligned magnetic field, it could only provide a crude approximation to any physically possible flow starting with a transverse magnetic field.

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Local Similarity Expansions of the Boundary-Layer Equations

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1. Formulation of the Asymptotic Expansion

CONSIDER the incompressible boundary-layer equation

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta(\xi)\{1 - (f_\eta)^2\} = 2\xi\{f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}\} \quad (1)$$

and its boundary conditions

$$f(\xi, 0) = f_\eta(\xi, 0) = 0 \qquad f_\eta(\xi, \infty) = 1 \quad (2)$$

that have been derived in Ref. 1, for instance. Introducing the inversion wherein the independent variables become (β, η),[1] Eqs. (1) and (2) may be written as

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta\{1 - (f_\eta)^2\} = \epsilon(\beta)\{f_\eta f_{\beta\eta} - f_\beta f_{\eta\eta}\} \quad (3)$$

$$f(\beta, 0) = f_\eta(\beta, 0) = 0 \qquad f_\eta(\beta, \infty) = 1 \quad (4)$$

where

$$\epsilon(\beta) = 2\xi\beta'(\xi) = 2\xi(\beta)/\xi'(\beta) \quad (5)$$

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